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RADIAL LIMIT SETS ON THE TORUS

BY

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ABSTRACT. Let U^N denote the unit polydisc and T^N the unit torus in the space of N complex variables. A subset A of T^N is called an (RL)-set (radial limit set) if to each positive continuous function ρ on T^N, there corresponds a function f in $H^\infty(U^N)$ such that the radial limit $|f|^*$ of the absolute value of f equals ρ, a.e. on T^N and everywhere on A. If $N > 1$, the question of characterizing (RL)-sets is open, but two positive results are obtained. In particular, it is shown that T^N contains an (RL)-set which is homeomorphic to a cartesian product $K \times T^{N-1}$, where K is a Cantor set. Also, certain countable unions of "parallel" copies of T^{N-1} are shown to be (RL)-sets in T^N. In one variable, every subset of T is an (RL)-set; in fact, there is always a zero-free function f in $H^\infty(U)$ with the required properties. It is shown, however, that there exist a circle $A \subset T^2$ and a positive continuous function ρ on T^2 to which correspond no zero-free f in $H^\infty(U^2)$ with $|f|^* = \rho$ a.e. on T^2 and everywhere on A.

1. **Introduction.** To each bounded, nonnegative function ρ on the unit circle T with $\log \rho \in L^1(T)$, there corresponds a bounded holomorphic function f on the unit disc U for which the radial limit $|f|^*$ of the absolute value of f equals ρ a.e. on T [3, p. 54]. It is known that this result does not generalize to the unit polydisc U^N in the space of N complex variables. However, one positive result due to Rudin [3, p. 55] asserts that if ρ is positive, bounded and lower semicontinuous on the unit torus T^N, there exists a function f in $H^\infty(U^N)$ with $|f|^* = \rho$ a.e. on T^N. In this paper, a modification of Rudin's construction will be used to obtain more precise information about the sets on which the equality $|f|^* = \rho$ is satisfied. In particular, the following related class of sets will be considered.

Definition. *A subset A of T^N is called an* (RL)*-set* (*radial limit set*) *if to each positive continuous function ρ on T^N, there corresponds a function f in $H^\infty(U^N)$ with $|f|^* = \rho$ a.e. on T^N and everywhere on A.*

In one variable, every subset of T is an (RL)-set. Indeed, if f is the outer function

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$$f(z) = \exp\left\{\int_T \frac{w+z}{w-z} \log \rho(w)\, dm(w)\right\}, \tag{1}$$

the continuity of ρ implies that $|f|^*(w) = \rho(w)$ for all $w \in T$. In several variables the question of characterizing (RL)-sets is open. However, in §2 of this paper, two types of (RL)-sets will be identified. In §3, some differences between one and several variables are discussed regarding the possibility of choosing a zero-free function f in the definition of (RL)-sets.

2. **Construction of radial limit sets.** The purpose of this section is to prove the following theorems which identify two types of (RL)-sets.

Theorem 1. *Let $a = (a_1, \cdots, a_N)$ be a point in Z^N with $a_j > 0$ for $1 \le j \le N$, and $\{p_k\}$ a sequence of complex numbers with $|p_k| = 1$.*

If $A_k = \{w \in T^N: w^a = p_k\}$, for $k = 1, 2, \cdots$, then $A = \bigcup A_k$ is an (RL)-set in T^N.

(As usual, Z^N denotes the space of lattice points $a = (a_1, \cdots, a_N)$ where each a_j is an integer. If $w = (w_1, \cdots, w_N)$ is in T^N, w^a stands for the monomial $w_1^{a_1} \cdots w_N^{a_N}$.)

Theorem 2. *Suppose K is the usual "middle-third" Cantor set on $[0, 1]$, and $S = \phi(K)$ where*

$$\phi(t) = -\exp(2\pi i t), \qquad (0 \le t \le 1).$$

If $A = \{w \in T^N: w_1 w_2 \cdots w_N \in S\}$, then A is an (RL)-set in T^N.

In Theorem 1, each set A_k consists of a finite number of "parallel" $(N-1)$-dimensional tori. Hence, the theorem asserts that any countable union of copies of T^{N-1} which are parallel to $\{w \in T^N: w^a = 1\}$ is an (RL)-set in T^N. Theorem 2 says that T^N contains an (RL)-set which is topologically the cartesian product of a Cantor set and an $(N-1)$-dimensional torus.

The first lemma is essentially Rudin's "modification theorem" [3, Theorem 2.4.2] upon which the construction of (RL)-sets will be based. Since the conclusion of the lemma is somewhat more detailed than Rudin's original version, a proof will be sketched.

As in [3, Chapter 2], $RP(T^N)$ will be the class of all complex Borel measures μ on T^N whose Poisson integral $P[d\mu]$ is the real part of a holomorphic function in U^N. RP-measures are characterized by the vanishing of their Fourier coefficients outside the positive and negative cones of Z^N.

If Q is a trigonometric polynomial on T^N, $\deg(Q)$ will denote the smallest positive integer d such that the Fourier coefficient $\hat{Q}(a)$ vanishes whenever $a = (a_1, \cdots, a_N) \in Z^N$ with $|a_j| > d$ for some j.

Lemma 1. *Suppose $\beta \in T^N$ and $s \in Z^N$ with $s_j > 0$ for $1 \le j \le N$. Let $E = \{w \in T^N: w^s = 1\}$, and $F = \beta E = \{(\beta_1 w_1, \cdots, \beta_N w_N): w \in E\}$. Let ν denote the Haar measure for the compact topological group E, and let μ be the translation of ν to the coset F; i.e., $\mu(A) = \nu(\bar{\beta} A)$. If Q is a nonnegative trigonometric polynomial on T^N with $\deg(Q) < s_j$ for $1 \le j \le N$, then*

(a) $Q - Qd\mu \in RP(T^N)$,

(b) $\hat{Q}(0) = (Qd\mu)\hat{}(0)$, *and*

(c) $\|Qd\mu\| = \|Q\|_1$.

Proof. For $a \in Z^N$, the Fourier coefficients of μ and ν are related by

$$\hat{\mu}(a) = \bar{\beta}^a \hat{\nu}(a). \tag{2}$$

The function $\bar{w}^a$ is a character on E and is identically 1 on E if and only if $a = ks$ for some integer k. Since ν is the Haar measure for E, it follows from (2) that

$$\hat{\mu}(a) = \begin{cases} \bar{\beta}^a & \text{if } a = ks \text{ for some } k \in Z, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Let $Y_N = Z_+^N \cup (-Z_+^N)$ where Z_+^N is the positive cone of all $a \in Z^N$ with $a_j \ge 0$ for $1 \le j \le N$. If Q is a nonnegative trigonometric polynomial on T^N, $\hat{Q}(a) = 0$ except for a in some finite set $X \subset Z^N$. Thus

$$0 \notin X + ks \subset Y_N, \qquad \text{for } k = \pm 1, \pm 2, \cdots, \tag{4}$$

whenever $s_j > \deg(Q)$ for $1 \le j \le N$.

It follows from (3) that

$$(Qd\mu)\hat{}(a) = \sum_{n \in X} \hat{Q}(n)\hat{\mu}(a-n) = \sum_{k=-\infty}^{\infty} \hat{Q}(a-ks)\bar{\beta}^{ks}. \tag{5}$$

If $a \notin Y_N$ and $k \neq 0$, (4) implies $a - ks \notin X$ so that $\hat{Q}(a - ks) = 0$. Hence, by (5), $(Qd\mu)\hat{}(a) = \hat{Q}(a)$ for all $a \notin Y_N$, which says that $Q - Qd\mu$ is in $RP(T^N)$.

Finally, it follows from (4) and (5) that

$$(Qd\mu)\hat{}(0) = \hat{Q}(0), \tag{6}$$

while $Q \ge 0$ implies $\|Qd\mu\| = (Qd\mu)\hat{}(0)$ and $\hat{Q}(0) = \|Q\|_1$. Hence, by (6), $\|Qd\mu\| = \|Q\|_1$, and the proof is complete.

Lemma 2. Let β, s, *and* μ *be defined as in Lemma* 1. *If $r = (r_1, \cdots, r_N)$ with $0 < r_j < 1$, and $w = (w_1, \cdots, w_N) \in T^N$, then the Poisson integral of μ is given by*

$$P[d\mu](rw) = \mathcal{P}((rw\bar{\beta})^s),$$

where $\mathcal{P}$ is the Poisson kernel in one variable,

$$\mathcal{P}(\zeta) = \operatorname{Re}[(1+\zeta)/(1-\zeta)] \qquad (\zeta \in U),$$

and

$$(rw\bar{\beta})^s = \prod_{n=1}^{N} (r_n w_n \bar{\beta}_n)^{s_n}.$$

Proof. The familiar series expansion [3, p. 17] for the Poisson kernel is combined with (3) to give

$$P[d\mu](rw) = \sum_{k=-\infty}^{\infty} \bar{\beta}^{ks} r_1^{|k|s_1} \cdots r_N^{|k|s_N} w^{ks}$$

$$= \operatorname{Re}\{[1 + (rw\bar{\beta})^s]/[1 - rw\bar{\beta})^s]\}.$$

The next lemma follows immediately from Lemma 2 and well-known properties of the Poisson kernel in one variable [2, p. 224].

Lemma 3. *Suppose β, s, μ, r and w are defined as before, and let*

$$\Gamma(\delta) = \{e^{i\theta}: 2\pi\delta \le \theta \le 2\pi(1-\delta)\},$$

and

$$M(\delta) = \sup\{\mathcal{P}(Re^{i\theta}): 0 < R < 1 \text{ and } e^{i\theta} \in \Gamma(\delta)\}.$$

Then,

(a) $P[d\mu](rw) \le M(\delta) < \infty$ *if* $0 < \delta < \frac{1}{2}$ *and* $(\omega\bar{\beta})^s \in \Gamma(\delta)$, *and*

(b) $\lim_{r\to 1} P[d\mu](rw) = 0$ *whenever* $(w\bar{\beta})^s \neq 1$.

Lemma 4. *Suppose $p_1, \cdots, p_n$ and $q_1, q_2, \cdots$ are points of T, and let Γ be a nondegenerate arc on T. To each number $\eta > 0$, there corresponds an integer $d > \eta$ and a point $\gamma \in T$ for which*

(a) $(p_k\gamma)^d \in \Gamma$ *for* $1 \le k \le n$, *and*

(b) $1 \notin \{(q_j\gamma)^d: 1 \le j < \infty\}$.

Proof. A classical theorem of Dirichlet [4, Volume I, p. 235] implies that for $\epsilon > 0$, there exists an integer $d > \eta$ for which $|1 - (p_k)^d| < \epsilon$ for $1 \le k \le n$. It follows that if ϵ is sufficiently small, there is an open arc Λ on T such that $(p_k)^d\lambda \in \Gamma$ whenever $\lambda \in \Lambda$. Since there are uncountably many points in T with $\gamma^d \in \Lambda$, such a point can be chosen so that none of the points $(q_j\gamma)^d$ (for $1 \le j < \infty$) is equal to 1.

The proof of Rudin's boundary value theorem [3, Theorem 3.5.3] can now be modified to establish Theorem 1.

Proof of Theorem 1. Let ρ be a positive continuous function on T^N and assume without loss of generality that $\log\rho > 0$. Choose nonnegative trigonometric polynomials Q_n on T^N such that $\log\rho = \sum_{n=1}^{\infty} Q_n$ on T^N, and, for $n = 1, 2, \cdots$,

$$\|Q_n\|_\infty \le 2^{1-n}\|\log\rho\|_\infty. \tag{7}$$

Fix δ with $0 < \delta < \frac{1}{2}$. For each $n = 1, 2, \cdots$, Lemma 4 implies that there exist an integer $d_n > \deg(Q_n)$ and a point $\gamma_n \in T$ such that

(8) $$(p_k \gamma_n)^{d_n} \in \Gamma(\delta) \quad \text{for } 1 \le k \le n,$$

and

(9) $$1 \notin \{(p_j \gamma_n)^{d_n}: 1 \le j < \infty\}.$$

Let $s_n = d_n a \in Z^N$ and choose $\beta_n \in T^N$ with $(\bar{\beta}_n)^a = \gamma_n$. As in Lemma 1, let $E_n = \{w \in T^N: w^{s_n} = 1\}$, $F_n = \beta_n E_n$, ν_n the Haar measure for E_n, and μ_n the translation of ν_n to the coset F_n. Since $d_n > \deg(Q_n)$, it follows from Lemma 1 that

$$Q_n - Q_n d\mu_n \in RP(T^N) \quad \text{for } n = 1, 2, \cdots,$$

and

$$\|Q_n d\mu_n\| = \|Q_n\|_1 \quad \text{for } n = 1, 2, \cdots.$$

Let $d\sigma_n = Q_n d\mu_n$. The trigonometric polynomials Q_n are nonnegative so that

$$\sum \|\sigma_n\| = \sum \|Q_n\|_1 = \int \sum Q_n = \int \log \rho < \infty,$$

and the series $\Sigma \sigma_n$ converges in total variation norm to a positive measure σ. Since each σ_n is singular (with respect to the Haar measure of T^N), so is σ. Moreover, if α lies outside the union of the positive and negative cones of Z^N, then

$$\hat{\sigma}(\alpha) = \sum \hat{\sigma}_n(\alpha) = \sum \hat{Q}_n(\alpha) = (\log \rho)\hat{}(\alpha),$$

so that $\log \rho - d\sigma$ is in $RP(T^N)$. In particular, there exists a holomorphic function g on U^N with

$$\text{Re}[g] = P[\log \rho - d\sigma].$$

Define $f = e^g$. Clearly, f is in $H^\infty(U^N)$ since $\log \rho$ is bounded above and $\sigma > 0$. Also, $|f|^* = \rho$ a. e. on T^N. In fact, the continuity of ρ implies

$$\lim_{r \to 1} P[\log \rho](rw) = \log \rho(w), \qquad \text{for all } w \in T^N,$$

hence $|f|^*(w) = \rho(w)$ if and only if

(10) $$\lim_{r \to 1} P[d\sigma](rw) = 0.$$

Thus, it remains to show that s_n and β_n have been chosen so that (10) holds for all $w \in A$.

If $w \in A_k$, then $w^a = p_k$ and it follows from the choice of s_n and β_n that

(11) $$(w\bar{\beta}_n)^{s_n} = (p_k \gamma_n)^{d_n}$$

for $n = 1, 2, \cdots$. Hence (8) implies

(12) $$(w\bar{\beta}_n)^{s_n} \in \Gamma(\delta) \quad \text{for } w \in A_k \text{ and } n \geq k,$$

while (9) gives

(13) $$(w\bar{\beta}_n)^{s_n} \neq 1 \quad \text{for } w \in A \text{ and } n = 1, 2, \cdots.$$

Since

(14) $$P[d\sigma_n](rw) \leq \|Q_n\|_\infty P[d\mu_n](rw),$$

it now follows from (7), (12), and Lemma 3 that

$$P[d\sigma_n](rw) \leq 2^{1-n} M(\delta) \| \log \rho \|_\infty$$

for all $w \in A_k$ and $n \geq k$. Hence, for each $w \in A_k$, the series $\sum_{n=1}^\infty P[d\sigma_n](rw)$ converges uniformly in r for $0 < r < 1$, and so

(15) $$\begin{aligned}\lim_{r\to 1} P[d\sigma](rw) = \lim_{r\to 1} P[d\sigma_n](rw) &= \sum \lim_{r\to 1} P[d\sigma_n](rw) \\ &\leq \sum 2^{1-n} \|\log \rho\|_\infty \lim_{r\to 1} P[d\mu_n](rw).\end{aligned}$$

Finally, (13) and Lemma 3 imply that for each $k = 1, 2, \cdots$, $\lim_{r\to 1} P[d\mu_n](rw) = 0$ if $w \in A_k$ and $n = 1, 2, \cdots$, so that by (15), $\lim_{r\to 1} P[d\sigma](rw) = 0$ for all $w \in A$, and the proof is complete.

Proof of Theorem 2. Let ρ be continuous on T^N with $\log \rho > 0$, and choose nonnegative trigonometric polynomials Q_n such that $\log \rho = \sum Q_n$, and $\|Q_n\|_\infty \leq 2^{1-n} \|\log \rho\|_\infty$. For each $n = 1, 2, \cdots$, choose an integer k_n such that $3^{k_n} > \deg(Q_n)$ and let $d_n = 3^{k_n}$. Let $E_n = \{w \in T^N : (w_1 w_2 \cdots w_N)^{d_n} = 1\}$, $\nu_n =$ Haar measure for E_n, and $d\sigma_n = Q_n d\nu_n$. If σ and f are now defined as in the proof of Theorem 1, it remains to show only that $\lim_{r\to 1} P[d\sigma](rw) = 0$ for all $w \in A$. This will follow exactly as in Theorem 1 from the following estimate:

(16) $$P[d\nu_n](rw) \leq M(1/6) \quad \text{for } w \in A,\ 0 < r < 1, \text{ and } n = 1, 2, \cdots,$$

where $M(1/6)$ is the supremum defined in Lemma 3.

To verify (16), observe that $\lambda \in S$ if and only if

(17) $$\lambda^{3^k} \in \Gamma(1/6) \quad \text{for each } k = 0, 1, 2, \cdots.$$

If w is in A, then $w_1 w_2 \cdots w_N$ is in S; in particular, by (17), $(w_1 w_2 \cdots w_N)^{d_n} \in \Gamma(1/6)$ for $n = 1, 2, \cdots$, and (16) follows from Lemma 3.

3. **Zero-free functions.** In one variable, the unit circle is an (RL)-set. In fact, the function (1) corresponding to the positive continuous function ρ on T has the additional property that it never vanishes in U. Whether the torus T^N is also an (RL)-set when $N > 1$ is an open question. However, the next theorem shows that in general the possibility of choosing a zero-free function in the definition

of (RL)-sets does not extend to several variables.

Theorem 3. *Suppose* ρ *is a positive continuous function on* T^N *and* f *a zero-free function in* $H^\infty(U^N)$ *with* $|f|^*(w) = \rho(w)$ *for all* $w \in T^N$. *Then* $\log \rho$ *is in* $RP(T^N)$.

Definition. *If* f *is a function on* U^N *and* $w \in T^N$, *the "slice function"* f_w *is defined on the unit disc by*

$$f_w(\lambda) = f(\lambda w) \qquad (\lambda \in U).$$

Proof of Theorem 3. Let f be any function in $H^\infty(U^N)$ with $|f^*|$ identically equal to ρ on T^N. For each $w \in T^N$, the slice function f_w is in $H^\infty(U)$ and $|f_w|$ has radial limits satisfying

$$|f_w|^*(\lambda) = \rho(\lambda w) > 0 \quad \text{for all } \lambda \in T.$$

Since the radial limit of a nonconstant singular inner function on U must vanish at some point of T, it follows that the inner factor of f_w is a Blaschke product [1, Chapter 5]. Hence, for each $w \in T^N$, the least harmonic majorant of $\log |f_w|$ is the Poisson integral $P[\log |f_w|^*]$. This implies, by [3, Theorem 3.3.6], that $P[\log \rho]$ is the least N-harmonic majorant of $\log |f|$ in U^N.

Now if f is never zero in U^N, $\log |f|$ is its own least N-harmonic majorant. Hence $\log |f| = P[\log \rho]$ and it follows [3, p. 73] that $\log \rho \in RP(T^N)$.

The final theorem illustrates more dramatically the difference between the situations in one and several variables. In particular, it implies that if A is the circle $\{(\zeta, \zeta): |\zeta| = 1\}$ in T^2, there exists a positive continuous function ρ on T^2 to which there corresponds no zero-free $f \in H^\infty(U^2)$ with $|f|^* = \rho$ a.e. on T^2 and everywhere on A.

Theorem 4. *Suppose* f *is a function in* $H^\infty(U^2)$ *with*

$$|f|^*(\zeta, \zeta) = 1 \quad \textit{for all } \zeta \in T, \tag{18}$$

and such that $f(\lambda, \lambda)$ *never vanishes for* $\lambda \in U$. *Then* $\|f_w^*\|_\infty \geq 1$ *for all* $w \in T^N$ (*where* $\|f_w^*\|_\infty$ *is the essential supremum of* $|f_w|^*$ *on* T).

Proof. Let $F(\lambda) = f(\lambda, \lambda)$ for $\lambda \in U$. Then $F \in H^\infty(U)$, F has no zeros in U, and by (18), $|F|^* = 1$ everywhere on T. In particular, F is a singular inner function. But, the radial limit of a nonconstant singular inner function must vanish at some point of T. So F must be constant and, in particular,

$$|f(0, 0)| = |F(0)| = 1. \tag{19}$$

Now suppose $\|f_w^*\|_\infty < 1$ for some $w \in T^N$. Since $f_w \in H^\infty(U)$, it follows that $|f_w(\lambda)| < 1$ for all $\lambda \in U$. In particular, $|f(0, 0)| = |f_w(0)| < 1$, which contradicts (19).

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